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A Remark on the Notion of Robust Phase Transitions

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We point out that the high-q Potts model on a regular lattice at its transition temperature provides an example of a nonrobust—in the sense recently proposed by Pemantle and Steif—phase transition.

KEY WORDS: Robust phase transitions; Potts models.

In a recent paper[18] about phase transitions of spin models on general trees, Pemantle and Steif introduced a distinction between the notions of ordinary and "robust" phase transitions. In their tree calculations, this notion of robust phase transitions was argued somehow to be more natural. They proposed as an interesting question to investigate this distinction also for regular lattices, and in particular the possible occurrence of non-robust phase transitions for spin models on regular lattices.

Here I point out that at the transition temperature of a high-q Potts model on a regular lattice the phase transition is non-robust, as a consequence of the occurrence of a first-order transition in the temperature variable.

A phase transition (for Ising, Potts or n-vector models) is called robust,[18] if weakening the bonds at the boundary of some large volume to an arbitrarily small (but positive) strength in some non-symmetric state does not influence the non-symmetric character of the spin distribution at the origin, in the limit where this boundary moves to infinity. By this construction one can interpolate between pure and free boundary conditions.

In the Ising model on a regular lattice robustness of the phase transition, when it occurs, follows from a result of Lebowitz and Penrose,[15] that for all temperatures weak positive boundary fields in the thermodynamic

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limit are equivalent to plus boundary conditions, in the sense that both induce the same pure (plus) Gibbs measure.

This is in fact not so surprising, as below the critical temperature free (symmetric) boundary conditions give rise to a symmetric mixture of pure ordered Gibbs measures, and these mixtures are unstable for even a small asymmetry at the boundary. After all, there are non-trivial—bounded—observables at infinity by which one can change the symmetric weight distribution of a mixture into an asymmetric one, and the interpretation of such observables at infinity as uniformly bounded, that is, quite weak, boundary terms has been known for a considerable time,[17] (see also ref. 4 for related arguments).

Indeed, let h be a non-trivial observable at infinity, such as exists for (and only for) a mixed Gibbs measure, say μ, then

$$\mu^h(\cdot) = \frac{\mu(\exp h \times \cdot)}{\mu(\exp h)} \neq \mu \tag{1}$$

Thus the Gibbs measure (Gibbsian for the same interaction as μ) μ^h which is obtained formally by adding the bounded term h to the Hamiltonian, is different from μ.

Moreover, in the references mentioned above it is described how μ^h can be approximated by a sequence of measures of the form μ^{h_n} where the h_n form a uniformly bounded sequence of boundary terms associated to a sequence of increasing volumes.

Thus it is not too surprising that breaking the symmetry in the boundary conditions in the manner described before, in which a small strength of a boundary field is multiplied by the size of the boundary, has a more drastic effect because this sequence of boundary terms diverges, instead of staying uniformly bounded. Indeed, in the Ising model, according to the Lebowitz–Penrose result it immediately drives the state to be extremal.

We remark as an aside that the notion of robustness is intimately linked with the single-spin space and the Hamiltonian having a symmetry which gets broken by the weak boundary field.

In the high-q Potts case at the transition temperature, however, in contrast to the low-temperature Ising case, there exists another Gibbs measure having the Potts permutation symmetry, beyond the symmetric mixture of the ordered measures. This is the disordered state, which can be obtained by imposing free boundary conditions, and which is pure, moreover.[13, 14, 3]

It takes more "boundary (free) energy" to move from one pure state to another than from a mixed state to one of its pure components. Intuitively put, if one thinks of the pure states as valleys in some (free) energy landscape, to go from one pure state to another, the system has to overcome a free energy barrier of boundary size. This represents the cost of inserting a droplet of another (here an ordered) phase. When the boundary bonds are too weak, even if the state outside Γ is ordered, they can't provide sufficient energy to cross this barrier between disordered and ordered phase, and favor order inside Γ.

On the other hand, mixed states are on top of this free energy barrier, and they can be easily pushed off from there.

To be a bit more precise we now adapt the definition of Pemantle and Steif to a regular lattice. (For the precise definitions, background and general notions of Gibbs measure theory we refer to refs. 9 or 19.)

Denote by $\mu^{+}_{J, \varepsilon, \Gamma}$ the infinite-volume measure which is obtained from the pure plus measure (we call the first of the q Potts-states also "plus" here by abuse of terminology) by multiplying the bond strengths J in some contour Γ by ε. Γ is a set of bonds (or edges) such that their dual bonds (plaquettes) form a closed (hyper-)surface separating the inside and outside of Γ. It plays the role of the "cutset" of ref. 18. The choice of the plus measure induces an effective boundary term, favoring order in the plus direction, at the boundary Γ of the volume $Int(\Gamma)$, the strength of which is bounded above by $2d \times \varepsilon \times \Gamma$.

Remark. We remind the reader that the (infinite-volume) plus measure is non-symmetric, due to the assumption of existence of a phase transition. Thus we have taken a first thermodynamic limit already, and (by the DLR-equations) the marginal measure to the configuration-space determined by the spins in the interior of Γ is an average over measures with different boundary conditions, all with weak boundary bonds, averaged with respect to this non-symmetric infinite-volume measure $\mu^{+}_{J, \varepsilon, \Gamma}$. As all boundary bonds in Γ are weak, this means that one can obtain the same measure by taking a suitable weak boundary term added to the finite-volume Hamiltonian.

Definition. A phase transition of a nearest neighbor Potts or n-vector model is robust if for the marginal measure to the single-site space at the origin for each positive $\varepsilon \in (0, 1]$, at least for some subsequence of increasing contours Γ_n whose interiors will finally include each finite volume,

$$\lim_{\Gamma_n \to \infty} \mu^{+}_{J, \varepsilon, \Gamma_n} \neq \mu^{free}_{J} \tag{2}$$

Our above discussion can be summarized in the following

Theorem. For the q-state Potts model on Z^d, d at least 2, with q high enough, at the transition temperature the phase transition is non-robust.

Proof. We claim that for small enough ε the above inequality (2) does not hold.

There are different ways one might approach a proof. We will sketch here how one can adapt the Fortuin–Kasteleyn random-cluster representation Pirogov–Sinai contour arguments of ref. 14 to obtain one. In this random-cluster representation (for a detailed description of the random-cluster representation for Potts models, see for example refs. 8, 7, 1, 11, and 10), one considers an associated correlated edge-percolation model, in which in a finite volume Λ the probability of an edge configuration $\eta \in (0, 1)^{B(\Lambda)}$ is given by:

$$\mu^{\Lambda}(\eta) = \frac{1}{Z} \prod_{e \in B(\Lambda)} p_e^{\eta_e}(1-p_e)^{1-\eta_e}\, q^{C(\eta)} \tag{3}$$

where $C(\eta)$ denotes the number of occupied connected clusters in the configuration η, and

$$p_e = 1 - \exp(-J_e) \tag{4}$$

with J_e the bond strength along edge e.

Percolation in the random-cluster model occurs if and only if there is long-range order in the associated spin model.

First we notice that the finite-volume measures obtained by taking wired boundary conditions outside a volume Λ_m containing Γ decrease, in FKG-sense, as the Λ_n grow, and each of them FKG-dominates the marginal on $\mathrm{int}(\Gamma)$ of the infinite-volume wired measure with "weak" bonds in Γ. The wired boundary conditions correspond to having all edges occupied outside the region, or, in spin language, to having all spins aligned (for instance in the plus configuration). In particular this infinite-volume wired state is associated to the measure $\mu^{+}_{J, \varepsilon, \Gamma}$.

These observations have several implications:

(i) The limit in the left-hand side of (2) always exists (in the weak sense). We emphasize that it is, in fact, a double limit formed by first taking for each Γ_n wired boundary conditions outside an increasing sequence of volumes Λ_m containing (the for the moment fixed) Γ_n, (this limit exists as it is a limit of FKG-decreasing measures), and then taking the limit $\Gamma_n \to \infty$.

(ii) If instead we take a "diagonal" limit $\Gamma_n \to \infty$ with wired boundary conditions outside Γ_n, we will see that we get a convergent sequence. Its limit is a measure that dominates any subsequence limit of the left-hand side of (2) (as for each Γ the element of the sequence dominates the corresponding element of the sequence in (i)). It turns out that it approaches the FKG-minimal state, that is, (for high q) the disordered state.

(iii) As the limit in (ii) is the FKG-minimal random-cluster state, so is the limit in (i).

We can take the Γ's to be the boundary of large squares in $d=2$, or (hyper-)cubes in higher dimensions, with the origin at the center.

In the first version of this paper I sketched how this weakly wired limit (ii) can be compared with (and can be shown to coincide with) the limit of measures with free boundary conditions through adaptation of existing proofs of the first-order Potts transition. I emphasized the arguments in ref. 3 (see Appendix), but also mentioned the random-cluster version. After submitting this version, R. Kotecký kindly informed me that a detailed version of the necessary contour analysis for the random-cluster version, essentially along this line, was worked out by I. Medved,[16] and included in his more general analysis of finite-size effects. In his terminology, having wired boundary conditions outside Γ and weak enough bonds in Γ, is an example of having "disordering boundary conditions." In his section (2.2) a range of ε which are disordering is determined.

With the above FKG-domination-argument, the statement of the non-robustness becomes indeed a corollary of his results. ▮

Although the proof thus rests on Medved's result, let me add some explanatory remarks, see also the Appendix. The proof goes essentially along the lines of ref. 14, once one notices that, because of the above, for small ε only a small fraction of the boundary edges in Γ (in the FK-representation[1, 8, 7, 11]) are occupied.

Thus the system acts essentially like the system with free boundary conditions outside Γ (up to a boundary term of order $\varepsilon J \times |\Gamma|$ which is small compared to the boundary free energy term necessary to induce the wired state on the inside of Γ which is of order $J \times \Gamma$). Differently put, the "weakly wired" boundary conditions do not influence the behavior in the infinite-volume—that is now infinite-Γ—limit, as compared to the free boundary conditions. The reason is that the Peierls contour estimate for the probability of finding an essentially ordered region inside Γ is exponentially small in $|\Gamma|$. Multiplying by a term $\exp(\varepsilon J \Gamma)$ does not qualitatively change this. At this point it is essential that q is large enough, and that one is at the transition temperature where the pure disordered state coexists with the q ordered ones. For the low temperature Ising model with free boundary conditions outside Γ the probability of finding the system inside Γ in an essentially plus configuration is $\frac{1}{2}$, and similarly for Potts models in the low temperature region this probability is $\frac{1}{q}$. These probabilities follow from symmetry, and are not obtained by contour estimates. It is here that the essential difference with the non-robustness example occurs.

For an earlier analysis of a related situation of changing "pure-state" to "almost-pure-state" boundary conditions in a more symmetric set-up see ref. 5.

Comment 1. The mechanism which causes the transition to be non-robust, is the fact that there is a first-order transition in temperature, such that at the transition temperature there is coexistence between a higher-entropy, lower-energy disordered state of higher symmetry, and a number of lower-entropy higher-energy states which are of lower symmetry. Thus in more complicated models, one expects the transition also to be non-robust whenever there is a first order transition in temperature, accompanied by the breaking of a symmetry on the low temperature side of the phase transition.

Comment 2. Although also for the high-q (in fact already for any value of q larger than 2) Potts model on trees the notions of ordinary and robust phase transitions do not coincide, and although, moreover, the high-q Potts model on trees, just as on regular lattices, typically shows a first-order transition in the temperature,[(18)] the interpretation of the distinction between ordinary and robust transitions seems somewhat different on trees. The separation between boundary terms and volume terms is more questionable on trees, so an interpretation in terms of a "boundary free energy" analysis does not seem possible. Indeed, on trees there can be coexistence of two ordered (the plus and minus) states with a disordered one, even for the Ising model, on a whole intermediate temperature interval.[(6, 2, 12)] For the Ising model on trees, however, Pemantle and Steif have showed that the occurrence of a phase transition and a robust phase transition always coincide.

APPENDIX

In this appendix I present a way to get an alternative derivation of Medved's result. In particular, I sketch how the arguments of the Pirogov–Sinai proof by Bricmont, Kuroda and Lebowitz,[(3)] (BKL) should be adapted to handle the case of weak boundary bonds.

We give, as in ref. 3, the argument for $d=2$. We compare only the weight of "boundary squares," that is squares inside Γ such that they touch at least one (and thus two or three) of the weak bonds, as all other computations are unchanged. The new element as compared to ref. 3 is that now an ordered boundary square will be a contour square (in the sense of BKL). Indeed, call a E_4' those sites in E_4 (that means, they touch four ordered bonds), which touch at least one weak bond. Such a site has at least $\frac{1}{4}\times(1-\varepsilon)$ less energy than it would have if none of the boundary bonds were weak, but still contributes zero entropy. This implies that even with ordered boundary conditions, an ordered boundary square with four E_4-sites has lower free energy compared to a disordered boundary square with four E_0-sites.

The definition of contours in terms of irregular—or contour—squares is then the same as in BKL.

Thus BKL Eq. (3.34), which estimates the partition function in volume Λ for all configurations compatible with a prescribed configuration of broken and unbroken bonds, is replaced by

$$Z(\Lambda \mid \underline{u}, \underline{b}) \leqslant q^{(E_0+E_4-E_4')} q^{(3/4+\varepsilon)(E_1+E_2+E_3+E_4')} \tag{5}$$

with Λ equal to int(Γ), and BKL Eq. (3.35), the inequality for the ratio of partition functions with or without the constraint that contour γ is present in volume Λ with boundary condition ω, which expresses that the Peierls condition applies,

$$Q(\gamma \mid \Lambda, \omega) \leqslant 2^{2\,|\gamma|} q^{-(C/4)\,|\gamma|} \tag{6}$$

still holds, but with a slightly worse constant C.

(As I already used the symbol Γ, I slightly changed the notation of ref. 3 to let γ denote a contour).

For a central square to be ordered, it needs to be surrounded by a contour, thus, at sufficiently high q and small enough (dependent on q) ε, this is of low probability, uniformly in the size of the enclosing boundary.

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